

DESCENDENT BOUNDS FOR EFFECTIVE DIVISORS ON \overline{M}_g

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ABSTRACT. The slope of \overline{M}_g is bounded from below by $\frac{60}{g+4}$ via a descendent calculation.

1. SLOPE

Let \overline{M}_g be the moduli space of curves for $g \geq 2$. Let $\lambda \in A^1(\overline{M}_g)$ be the first Chern class of the Hodge bundle. Let $\delta_0 \in A^1(\overline{M}_g)$ be the class of the boundary divisor of irreducible nodal curves. For $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$, let $\delta_i \in A^1(\overline{M}_g)$ be the class of the corresponding reducible boundary divisor. We will consider *effective* divisors $D \subset \overline{M}_g$ of the form

$$(1) \quad [D] = \alpha\lambda - \beta_0\delta_0 - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \beta_i\delta_i, \quad \alpha, \beta_i \in \mathbb{Q}_{>0}.$$

Our goal is to find lower bounds for the slope $\frac{\alpha}{\beta_0}$ of D .

If $C \subset \overline{M}_g$ is a curve with no components contained in D , then

$$[C] \cdot [D] \geq 0.$$

If also $[C] \cdot \lambda > 0$ and C has no components contained in the boundary divisors, then we conclude

$$\frac{\alpha}{\beta_0} \geq \frac{[C] \cdot \delta_0}{[C] \cdot \lambda}.$$

Hence, such curves C provide lower bounds for the slope.

2. COTANGENT LINES

Let $\psi_i \in A^1(\overline{M}_{g,n})$ be the first Chern class of the cotangent line bundle

$$L_i \rightarrow \overline{M}_{g,n}$$

associated to the i^{th} marked point. The line bundles L_i are well-known to be nef via the Hodge index theorem for surfaces. Hence the class

$$(2) \quad \psi_1^{r_1} \cdots \psi_n^{r_n} \in A_1(\overline{M}_{g,n}), \quad \sum_{i=1}^n r_i = 3g - 3 + n - 1$$

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is a limit of effective curve classes which sweep $\overline{M}_{g,n}$. We can push the curve class (2) forward by

$$\epsilon : \overline{M}_{g,n} \rightarrow \overline{M}_g.$$

Therefore,

$$(3) \quad \frac{\int_{\overline{M}_{g,n}} \psi_1^{r_1} \cdots \psi_n^{r_n} \cdot \delta_0}{\int_{\overline{M}_{g,n}} \psi_1^{r_1} \cdots \psi_n^{r_n} \cdot \lambda}$$

is a lower bound for slopes of effective divisors D of the form (1) as long as the denominator is positive.

3. CALCULATION

We calculate the bound (3) in case $n = 1$ and $r_1 = 3g - 3$ by explicit evaluation of the two integrals.

Consider first the numerator. Using the normalization map of the irreducible divisor,

$$\int_{\overline{M}_{g,1}} \psi_1^{3g-3} \cdot \delta_0 = \frac{1}{2} \int_{\overline{M}_{g-1,3}} \psi_1^{3g-3} = \frac{1}{2} \int_{\overline{M}_{g-1,1}} \psi_1^{3g-5}.$$

The string equation is used in the last equality. The evaluation $\int_{\overline{M}_{g-1,1}} \psi_1^{3g-5} = \frac{1}{(24)^{g-1}(g-1)!}$ is well-known [2]. Hence

$$\int_{\overline{M}_{g,1}} \psi_1^{3g-3} \cdot \delta_0 = \frac{1}{2} \frac{1}{(24)^{g-1}(g-1)!}.$$

The denominator is more complicated to evaluate. The first step is to use the GRR equation of [1] for Hodge integrals,

$$(4) \quad \int_{\overline{M}_{g,1}} \psi_1^{3g-3} \cdot \lambda = \frac{B_2}{2} \int_{\overline{M}_{g,2}} \psi_1^{3g-3} \psi_2^2 - \frac{B_2}{2} \int_{\overline{M}_{g,1}} \psi_1^{3g-2} + \frac{B_2}{4} \int_{\overline{M}_{g-1,3}} \psi_1^{3g-3},$$

where $B_2 = 1/6$ is the second Bernoulli number. We have already seen how to evaluate the last two integrals. The first integral on the right side of (4) is evaluated using the L_1 Virasoro constraint [4],

$$\frac{15}{4} \int_{\overline{M}_{g,2}} \psi_1^{3g-3} \psi_2^2 = \frac{(6g-5)(6g-3)}{4} \int_{\overline{M}_{g,1}} \psi_1^{3g-2} + \frac{1}{2} \cdot \frac{1}{4} \int_{\overline{M}_{g-1,3}} \psi_1^{3g-3}.$$

Putting this together, we find

$$\int_{\overline{M}_{g,1}} \psi_1^{3g-3} \cdot \lambda = \left(\frac{(6g-5)(6g-3)}{180} - \frac{1}{12} \right) \frac{1}{(24)^g g!} + \frac{2}{45} \frac{1}{(24)^{g-1}(g-1)!}.$$

After taking the ratio, we obtain a simple exact evaluation of the bound,

$$\frac{\int_{\overline{M}_{g,1}} \psi_1^{3g-3} \cdot \delta_0}{\int_{\overline{M}_{g,1}} \psi_1^{3g-3} \cdot \lambda} = \frac{60}{g+4}.$$

Proposition. *The slope of $D \subset \overline{M}_g$ is always at least $\frac{60}{g+4}$.*

Taking the limit for large g , we find

$$\lim_{g \rightarrow \infty} \left(\frac{\int_{\overline{M}_{g,1}} \psi_1^{3g-3} \cdot \delta_0}{\int_{\overline{M}_{g,1}} \psi_1^{3g-3} \cdot \lambda} \right) \sim \frac{60}{g}.$$

Hence, we derive the asymptotic $\frac{1}{g}$ bound predicted experimentally in [3].

The method of [3] is the same, but the moving curves there are obtained from Hurwitz covers rather than descendent intersections. The combinatorics of Hurwitz covers makes exact analysis difficult, but computer calculations [3] predict the slope is always at least $\frac{576}{5g}$, which is not very different from the Proposition. The advantage of the descendent approach is the calculational simplicity.

4. OTHER BOUNDS

Low genus computations via Faber's program suggest the following property always holds.

Conjecture. For $g \geq 1$ and $\sum_{i=1}^n r_i = 3g - 3 + n - 1$

$$\frac{\int_{\overline{M}_{g,1}} \psi_1^{3g-3} \cdot \delta_0}{\int_{\overline{M}_{g,1}} \psi_1^{3g-3} \cdot \lambda} \geq \frac{\int_{\overline{M}_{g,n}} \psi_1^{r_1} \cdots \psi_n^{r_n} \cdot \delta_0}{\int_{\overline{M}_{g,n}} \psi_1^{r_1} \cdots \psi_n^{r_n} \cdot \lambda}.$$

If true, the bound of the Proposition is the best obtainable from descendent integrals on the moduli space of curves.

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